LETTER

Degrees-of-Freedom Based on Interference Alignment with Imperfect Channel Knowledge*

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SUMMARY The impact and benefits of channel state information (CSI) are analyzed in terms of degrees-of-freedom (DoFs) in a K-user interference network operating over time-selective channels, where the error variance of CSI estimation is assumed to scale with an exponent of the received signal-to-noise ratio (SNR). The original interference alignment (IA) scheme is used with a slight modification in the network. Then, it is shown that the DoFs promised by the original IA can be fully achieved under the condition that the CSI quality order, represented as a function of the error variance and the SNR, is greater than or equal to 1. Our result is extended to the case where the number of communication pairs, K, scales with the SNR, i.e., infinite K scenario, by introducing the user scaling order. As a result, this letter provides vital information to the system designer in terms of allocating training resources for channel estimation in practical cellular environments using IA.

key words: *channel state information (CSI), degrees-of-freedom (DoFs), error variance, interference alignment (IA), signal-to-noise ratio (SNR)*

1. Introduction

To suppress interference between users is an important problem in communication systems where multiple users share the same resources. Recently, interference alignment (IA) was introduced for fundamentally solving the interference problem when there are multiple communication pairs [1]. It was shown that the IA scheme can achieve the optimal degrees-of-freedom (DoFs), also known as capacity prelog factor, which is equal to $K/2$ in a K-user interference channel with time-varying channel coefficients. The basic idea of the scheme is to confine all the undesired interference from other communication links into a pre-defined subspace, whose dimension is the same as that of the desired signal space, thereby enabling all users to achieve one half of their available DoFs. The seminar work [1] has led to interference management schemes based on IA in various wireless network environments: multiple-input multiple-output (MIMO) interference network [2], [3], X network [4], [5], and cellular network [6].

However, all results in [1]–[4], [6] are based on perfect channel state information (CSI) of all (or local) network links at each transmitter, where perfect CSI is needed to achieve the optimal DoFs. The IA schemes thus face the practical challenge of obtaining the CSI, which is acquired at each transmitter by using either quantized feedback signaling or channel estimation through pilots in practical environments. Recently, the impact of imperfect CSI on the performance of IA has extensively been studied in [7]–[13]. Specifically, performance degradation was examined with respect to the achievable sum rates in time-varying MIMO channels [7]. In [8], it was shown that only the quantized CSI via limited feedback is needed to obtain full DoFs of $K/2$ for a frequency-selective setup, provided that it is beyond a certain level. This result was extended to the MIMO interference channel with multiple frequency slots [9] and the two-cell MIMO uplink channel [10]. On the other hand, other studies, [11]–[13], has examined the MIMO interference channel scenario with constant coefficients (i.e., time-invariant or narrow-band model). In [11], [12], the performance measures such as symbol error rate or achievable sum rates were quantified by characterizing the signal-to-interference-and-noise ratio (SINR) under imperfect CSI assumption. IA based on analog feedback [13] was also introduced in the frequency division duplexing systems where the reciprocity of the forward and reverse channels does not hold.

While a variety of IA schemes have been shown in the constant MIMO interference channel, the optimal DoFs for the model are still an open problem [2]. In this paper, we characterize how performance on the optimal DoFs, shown in [1], degrades in a K-user interference network operating over time-varying channels, where imperfect CSI is acquired at each node via channel estimation. In particular, under the network, we analyze the relationship between channel uncertainty and its effect on the DoFs. To our knowledge, such an attempt has never been done yet in the literature. We first define the CSI quality order, which is based on the assumption that the error variance of CSI estimation scales with an exponent of the received signal-to-noise ra-

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tio (SNR). Then, when the original IA scheme in [1] is used with a slight modification in the network, it is shown that if the error variance scales smaller than or as a certain exponent of the received SNR, or equivalently, the CSI quality order is greater than or equal to 1, then full DoFs are maintained. Moreover, it is seen that there exists a continuous trade-off between the achievable DoFs and the CSI quality order. We also examine the case where the number of communication pairs, K, scales with the SNR (i.e., infinite K condition).

The rest of this paper is organized as follows. In Sect. 2, we introduce the system and channel models. The IA scheme with imperfect CSI is described for multi-user interference networks in Sect. 3, and its achievability result is shown in terms of DoFs in Sect. 4. Finally, Sect. 5 summarizes this paper with some concluding remarks.

Throughout this paper, the superscripts T, H and $\dagger$ denote the transpose, conjugate transpose and pseudo-inverse, respectively, of a matrix (or a vector). $(x)^+$ denotes $\max\{0, x\}$, $[\cdot]_{ki}$ is the (k, i)-th element of a matrix, and $\doteq$ is referred to as the exponential equality. Unless otherwise stated, all logarithms are assumed to be to the base 2.

2. System and Channel Models

We consider the K-user interference channel, composed of K transmitters and K receivers, as in [1], where each receiver is interested only in traffic demands of the corresponding communication pair (transmitter). It is assumed that each node is equipped with a single antenna.

We assume frequency-flat and time-varying channel coefficients, where M symbols are transmitted over M time slots†. The term $\mathbf{H}^{[kj]} \in \mathbb{C}^{M\times M}$ denotes a diagonal matrix from the j-th transmitter to the k-th receiver, where $[\mathbf{H}^{[kj]}]_{nn}$ represents the n-th extension of the channel for $k, j \in \{1, \cdots, K\}$ and $n \in \{1, \cdots, M\}$. The channel (matrix) is assumed to be Rayleigh, whose diagonal elements have zero-mean and unit variance, and to be independent of different k, j, and time. Then, the channel output $\mathbf{y}^{[k]} \in \mathbb{C}^{M\times 1}$ at the k-th user's receiver is expressed as follows:

$$\mathbf{y}^{[k]} = \sqrt{\mathrm{SNR}} \sum_{j=1}^{K} \mathbf{H}^{[kj]}\mathbf{x}^{[j]} + \mathbf{z}^{[k]}, \tag{1}$$

where $k \in \{1, \cdots, K\}$ is the user index, $\mathbf{x}^{[k]} \in \mathbb{C}^{M\times 1}$ is the input signal vector at the k-th transmitter whose elements have unit variance, and $\mathbf{z}^{[k]} \in \mathbb{C}^{M\times 1}$ is the independent identically distributed (i.i.d.) and circularly symmetric complex additive white Gaussian noise (AWGN) vector at the k-th receiver with zero-mean and covariance matrix $\mathbf{I}_M$.

We model the imperfect CSI at both transmitter and receiver sides, $\hat{\mathbf{H}}^{[kj]}$, as

$$\mathbf{H}^{[kj]} = \hat{\mathbf{H}}^{[kj]} + \tilde{\mathbf{H}}^{[kj]}, \tag{2}$$

where $\tilde{\mathbf{H}}^{[kj]}$ denotes the CSI estimation error matrix whose elements have zero-mean and variance $\sigma^2_{\tilde{\mathbf{H}}}$, and is circularly symmetric complex Gaussian, where all (k, j) pairs are assumed to have the same variance††. Suppose that the estimation performance gets improved with increasing received SNR. In this case, if the CSI quality order α is defined as

$$\alpha = \lim_{\mathrm{SNR}\to\infty} -\frac{\log \sigma^2_{\tilde{\mathbf{H}}}}{\log \mathrm{SNR}}, \tag{3}$$

then the error variance $\sigma^2_{\tilde{\mathbf{H}}}$ is characterized by $\sigma^2_{\tilde{\mathbf{H}}} \doteq \mathrm{SNR}^{-\alpha}$, which is motivated by the mean square error on the CSI that polynomially decreases with increasing SNR in point-to-point links. For example, $\sigma^2_{\tilde{\mathbf{H}}} = 0$ indicates perfect CSI while no CSI is available when $\sigma^2_{\tilde{\mathbf{H}}} = 1$.

3. IA Based on Imperfect CSI

We basically use the IA scheme of [1] with a slight modification. In this section, we briefly address the main difference caused by the imperfect CSI assumption in the sense of designing pre- and post-processing matrices.

Suppose that message W_j is encoded at the j-th transmitter into $d^{[j]}$ independently encoded data streams $s_m^{[j]}$ sent along beamforming vectors $\mathbf{v}_m^{[j]} \in \mathbb{C}^{M\times 1}$ so that $\mathbf{X}^{[j]}$ is expressed as

$$\mathbf{X}^{[j]} = \sum_{m=1}^{d^{[j]}} s_m^{[j]}\mathbf{v}_m^{[j]} = \mathbf{V}^{[j]}\mathbf{s}^{[j]}, \tag{4}$$

where the $M \times d^{[j]}$-dimensional transmit beamforming matrix $\mathbf{V}^{[j]}$ is generated by using the estimated channel matrix $\hat{\mathbf{H}}^{[kj]}$ for all (k, j) pairs and is given by

$$\mathbf{V}^{[j]} = \left[\mathbf{v}_1^{[j]} \ \cdots \ \mathbf{v}_{d^{[j]}}^{[j]}\right], \tag{5}$$

and the input signal $\mathbf{s}^{[j]} \in \mathbb{C}^{d^{[j]}\times 1}$ at the j-th transmitter, composed of $d^{[j]}$ streams, is given by

$$\mathbf{s}^{[j]} = \left[s_1^{[j]} \ \cdots \ s_{d^{[j]}}^{[j]}\right]^T. \tag{6}$$

Here, according to the transmission strategy in [1], we select $d^{[j]}$ such that

$$\lim_{M\to\infty} \frac{\sum_{j=1}^{K} d^{[j]}}{KM} = \frac{1}{2} \tag{7}$$

for all $j \in \{1, \cdots, K\}$. Then from (1), the output signal $\mathbf{y}^{[k]}$ at the k-th receiver can be rewritten as

†The M channel uses can also be over frequency slots, or a time-frequency tuple if coding across both time and frequency is performed.

††As an example, consider time division duplexing (TDD) downlink systems, where the transmitters and receivers act as base stations (BSs) and user equipments, respectively, and the high capacity backhaul between BSs is used to exchange cross-link CSIs (e.g., $\hat{\mathbf{H}}^{[ki]}$ for the j-th transmitter's aspect where $i \neq j$ and $k = 1, \cdots, K$). It is then possible for each BS to estimate all the CSIs and thus to perform its transmit processing by utilizing the channel reciprocity in the TDD systems.

$$\begin{aligned}\mathbf{y}^{[k]} &= \sqrt{\mathrm{SNR}}\sum_{j=1}^{K}\mathbf{H}^{[kj]}\mathbf{V}^{[j]}\mathbf{s}^{[j]}+\mathbf{z}^{[k]}\\ &= \sqrt{\mathrm{SNR}}\left(\hat{\mathbf{H}}^{[kk]}+\tilde{\mathbf{H}}^{[kk]}\right)\mathbf{V}^{[k]}\mathbf{s}^{[k]}\\ &= \sqrt{\mathrm{SNR}}\sum_{\substack{j=1\\ j\neq k}}^{K}\left(\hat{\mathbf{H}}^{[kj]}+\tilde{\mathbf{H}}^{[kj]}\right)\mathbf{V}^{[j]}\mathbf{s}^{[j]}+\mathbf{z}^{[k]}\\ &= \sqrt{\mathrm{SNR}}\hat{\mathbf{H}}^{[kk]}\mathbf{V}^{[k]}\mathbf{s}^{[k]}+\underbrace{\sqrt{\mathrm{SNR}}\sum_{\substack{j=1\\ j\neq k}}^{K}\hat{\mathbf{H}}^{[kj]}\mathbf{V}^{[j]}\mathbf{s}^{[j]}}_{\text{The term that can be eliminated by IA}}\\ &\quad+\underbrace{\sqrt{\mathrm{SNR}}\sum_{j=1}^{K}\tilde{\mathbf{H}}^{[kj]}\mathbf{V}^{[j]}\mathbf{s}^{[j]}+\mathbf{z}^{[k]}}_{\text{Effective noise}}. \end{aligned} \tag{8}$$

Note that since $\mathbf{V}^{[j]}$ is designed based on $\hat{\mathbf{H}}^{[kj]}$ for all (k, j) pairs, the second term on the right-hand-side (RHS) of (8) is eliminated, whereas the sum of the third term, caused by interference misalignment, and the noise $\mathbf{z}^{[k]}$ will be treated as an effective noise.

At the k-th receiver side, a simple zero-forcing (ZF) filtering is performed by multiplying the post-processing matrix $\left(\hat{\mathbf{H}}^{[kk]}\mathbf{V}^{[k]}\right)^{\dagger} \in \mathbb{C}^{d^{[k]}\times M}$, and the resulting signal is given by

$$\begin{aligned}\left(\hat{\mathbf{H}}^{[kk]}\mathbf{V}^{[k]}\right)^{\dagger}\mathbf{y}^{[k]} &= \sqrt{\mathrm{SNR}}\mathbf{s}^{[k]}\\ &+\left(\hat{\mathbf{H}}^{[kk]}\mathbf{V}^{[k]}\right)^{\dagger}\left(\sqrt{\mathrm{SNR}}\sum_{j=1}^{K}\tilde{\mathbf{H}}^{[kj]}\mathbf{V}^{[j]}\mathbf{s}^{[j]}+\mathbf{z}^{[k]}\right).\end{aligned}$$

4. Analysis of Achievable DoFs

In this section, we show how the DoFs of the K-user interference network behaves according to the channel uncertainty.

4.1 Main Result

The total number of achievable DoFs is as follows:

$$\sum_{j=1}^{K} d^{[j]} = \lim_{\mathrm{SNR}\to\infty}\frac{R(\mathrm{SNR})}{\log \mathrm{SNR}}, \tag{9}$$

where $R(\mathrm{SNR})$ denotes the achievable sum rates. Since Gaussian is the worst additive noise, assuming it for the effective noise in (8) lower-bounds the sum rates, thereby yielding

$$R(\mathrm{SNR}) \geq \frac{1}{M}\sum_{j=1}^{K}\sum_{m=1}^{d^{[j]}}\log\left(1+\mathrm{SINR}_m^{[j]}\right), \tag{10}$$

where $\mathrm{SINR}_m^{[j]}$ represents a lower bound on the received SINR of each desired stream for all m and j. Now we are ready to show our main result in the network.

Proposition 1. *Suppose that for the K-user interference network over M time slots, IA is performed based on imperfect CSI with the CSI quality order α. Then, the total number of achievable DoFs approaches $\frac{K}{2}\min\{1,\alpha\}$ for large M, where $\alpha \in [0,\infty)$.*

Proof. The received SINR of the m-th stream at the k-th receiver is lower-bounded by

$$\begin{aligned}\mathrm{SINR}_m^{[j]} &= \frac{\mathrm{SNR}}{K\cdot \mathrm{SNR}\sigma_{\tilde{\mathbf{H}}}^2+1}\\ &\quad\cdot\left[\left(\left(\hat{\mathbf{H}}^{[kk]}\mathbf{V}^{[k]}\right)^H\left(\hat{\mathbf{H}}^{[kk]}\mathbf{V}^{[k]}\right)\right)^{-1}\right]_{m,m}^{-1}\\ &\doteq \frac{\mathrm{SNR}}{K\cdot \mathrm{SNR}^{1-\alpha}+1}\\ &\quad\cdot\left[\left(\left(\hat{\mathbf{H}}^{[kk]}\mathbf{V}^{[k]}\right)^H\left(\hat{\mathbf{H}}^{[kk]}\mathbf{V}^{[k]}\right)\right)^{-1}\right]_{m,m}^{-1}\\ &\doteq \min\{\mathrm{SNR},\mathrm{SNR}^{\alpha}\},\end{aligned} \tag{11}$$

where the first exponential equality holds since $\sigma_{\tilde{H}}^2 \doteq \mathrm{SNR}^{-\alpha}$. The second exponential equality holds comes from the fact that the term

$$\left[\left(\left(\hat{\mathbf{H}}^{[kk]}\mathbf{V}^{[k]}\right)^H\left(\hat{\mathbf{H}}^{[kk]}\mathbf{V}^{[k]}\right)\right)^{-1}\right]_{m,m}^{-1} \tag{12}$$

does not scale with the SNR. Therefore, using (10) and (11) in (9) results in

$$\sum_{j=1}^{K} d^{[j]} \geq \frac{\sum_{j=1}^{K} d^{[j]}\min\{1,\alpha\}}{M}, \tag{13}$$

thus yielding $\frac{K}{2}\min\{1,\alpha\}$ for large M due to the condition in (7), which completes the proof. □

This result is illustrated in Fig. 1(a). Note that there is no loss in terms of DoFs for the interference network even with imperfect CSI, compared to the perfect CSI scenario, as long as the CSI quality order α is greater than or equal to 1, i.e., the error variance $\sigma_{\tilde{\mathbf{H}}}^2$ scales slower than or as $1/\mathrm{SNR}$. For $\alpha \in [0, 1]$, it is seen that the number of achievable DoFs is linearly decaying with respect to decreasing α.

4.2 Extension to Infinite User Scenario

We now take into account the case where the number of

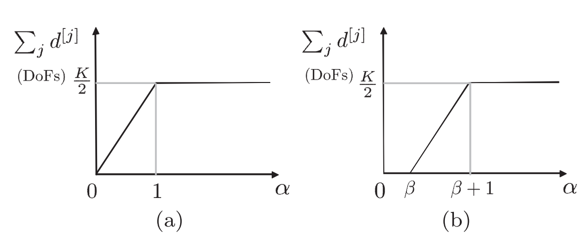

Fig. 1 The total numer of achievable DoFs. (a) Finite K case. (b) Infinite K case.

users, K, also scales with the received SNR, i.e., the case for infinitely many users. Let us introduce a parameter $\beta \geq 0$, called the user scaling order, satisfying $K \doteq \mathrm{SNR}^{\beta}$. The parameter β then controls how fast K scales with increasing SNR. This scenario is reasonable since as the received SNR increases, more communication pairs can be activated simultaneously. Then, as illustrated in Fig. 1(b), the total number of DoFs asymptotically achieves

$$\frac{K}{2}\min\{1, (\alpha-\beta)^{+}\}, \tag{14}$$

where $\alpha, \beta \in [0, \infty)$. This is because the lower bound $\mathrm{SINR}_m^{[j]}$ on the received SINR of each stream is given by

$$\begin{aligned}\mathrm{SINR}_m^{[j]} &\doteq \frac{\mathrm{SNR}}{\mathrm{SNR}^{1-\alpha+\beta}+1} \\ &\quad \cdot \left[\left(\left(\hat{\mathbf{H}}^{[kk]}\mathbf{V}^{[k]}\right)^{H}\left(\hat{\mathbf{H}}^{[kk]}\mathbf{V}^{[k]}\right)\right)^{-1}\right]_{m,m}^{-1} \\ &\doteq \min\{\mathrm{SNR}, \mathrm{SNR}^{\alpha-\beta}\} \end{aligned} \tag{15}$$

for all $j \in \{1, \cdots, K\}$ and $m \in \{1, \cdots, d^{[j]}\}$.

We now specify how the relationship between α and β affects the performance on the DoFs. From Fig. 1(b), if $\alpha \leq \beta$ (i.e., the number of users, K, scales faster than or as $\sigma_{\tilde{\mathbf{H}}}^{-2}$), then it follows that $\sum_j d^{[j]} = 0$. It is also seen that there is no loss on the DoFs, compared to the perfect CSI case, provided that $\alpha \geq \beta + 1$. Note that unlike the case of perfect CSI estimation (i.e., $\sigma_{\tilde{\mathbf{H}}}^{2} = 0$), the number of DoFs can be reduced with increasing K.

5. Conclusion

The relation between the CSI quality and the DoFs was analyzed in the K-user interference network with IA operating over time-varying channels. It was shown that DoFs promised by the original IA scheme can be fully achieved if the CSI quality order is greater than or equal to 1 under the finite K condition. Furthermore, it turned out that there is a continuous trade-off between the achievable DoFs and the CSI quality order α represented as a function of the error variance of CSI estimation and the SNR. The result was extended to the scenario where K increases with the SNR.

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